WAVE PROPAGATION IN PIEZOELECTRIC MEDIA OF HEXAGONAL SYMMETRY, AND THE EFFECTS OF A LATERAL EDGE

by

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Elastic wave propagation is considered in piezoelectric media of hexagonal symmetry. Having examined some of the implications of a hexagonal elastic symmetry, the piezoelectric infinite plate is considered, for which the permissible modes of propagation and the criterion of resonance are established. The modes of propagation and resonance requirements are then established for a plate rotated an arbitrary number of degrees from the crystallographic, or poled, axis.

The effects of a lateral edge are examined, indicating that an edge-associated mode can exist which dampens exponentially away from the edge, the problem degenerating rather quickly to the infinite plate case.

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I. INTRODUCTION

The problem of elastic wave propagation in isotropic plates was first considered by Lord Rayleigh in 1889.(1)* More recently Lamb(2) derived the period equation for dispersed wave propagation in an infinite plate, and Tolstoy and Usdin(3) and many others treated various aspects of wave propagation in a plate, such as the phase and group velocity dispersion, and characteristics of higher modes. Morse(4) compared the observed dispersion of compressional waves in a long beam and found results which agree well with those of an infinite plate when the largest dimension of the beam is taken as the plate thickness. Holden,(5) Bishop(6) and others considered finite geometries such as cylinders and beams.

The propagation of elastic waves in aeolotropic media has been less extensively studied. Sakadi(7) has considered elastic waves in most crystal systems and Musgrave(8) derived by variational calculus the aeolotropic wave equations as well as expressions for wave surfaces.

Problems in elastic wave propagation in piezoelectric material have been considered by Koga(9) who treated resonance in an infinite piezoelectric plate, by Cady,(10)(11) and Lawson(12) who considered the effects of electrode spacing on resonance of an infinite plate. Tiersten(13)(14) showed that the three roots to the wave equation corresponding to the longitudinal and two shear modes are in general all needed to satisfy the infinite plate boundary conditions but that each root is independent in

* Numbers correspond to References at the end of the paper.

the common hexagonal crystal system. Ekstein[15] has developed a perturbation method of obtaining approximate frequencies of a plate bounded by two normal planes which matches the experimental data of Sykes[16] except near flexural resonance.

Problems in elastic wave propagation are considerably more complex in a piezoelectric medium than in an isotropic elastic medium, due to the additive difficulties of an aeolotropic elastic structure and the piezoelectric effect which introduces extra terms into the equations of state. This would be, and in some respects certainly is, the case were it not for the fact that the boundary conditions which are physically most significant result in a considerable simplification of the mathematics, to the end that a reasonably complete solution of wave propagation in an infinite plate has been accomplished for both isotropic and piezoelectric media, while only modest inroads have been made into the finite geometries.

There are two aspects of wave propagation in a piezoelectric medium which are to be considered here. The first is an extension of the solution for an infinite plate to that of an infinite plate which has between the parallel electrode faces a piezoelectric material with a crystallographic, or z, axis at an arbitrary angle to the normal to the plate face (a "rotated" plate). The second is the effect of a lateral face (a "finite" plate). Each of these problems is treated only for media with hexagonal symmetry, and where appropriate, results will be evaluated for the most common ferroelectric ceramic, barium titanate, which has hexagonal elastic symmetry (as do most of the ferroelectric ceramics such as the lead-zirconate-titanate compounds).

II. NOTATION

A. The Piezoelectric Equations of State

Table I gives a summary of the notation to be used. Those symbols which are applicable are in accordance with the IRE Standards on Piezoelectric Crystals.[17]

The piezoelectric equations of state which are appropriate in a rationalized m.k.s. system are

$$T_{ij} = c_{ijkl} S_{kl} - e_{mij} E_m \qquad \text{(II - 1)}$$

$$D_i = e_{ikl} S_{kl} + \epsilon_{im} E_m \qquad \text{(II - 2)}$$

where,

$$S_{kl} = \tfrac{1}{2} (u_{k,l} + u_{l,k}) \qquad \text{(II - 3)}$$

$$E_k = -\varphi_{,k} \qquad \text{(II - 4)}$$

while the wave equations, and appropriate Maxwell equations are

$$T_{ij,i} = \rho u_{j,tt} \qquad \text{(II - 5)}$$

$$D_{i,i} = 0 \qquad \text{(II - 6)}$$

In all of the above, the tensor summation convention has been observed, as well as the use of a comma to designate the operation of taking a partial derivative with respect to the variable following the comma, thus $\frac{\partial D_i}{\partial x_i}$ becomes $D_{i,i}$.

B. Engineering Notation

Although it has been necessary to write out the equations (II - 1,2,3,5) in their full tensor form, it will prove convenient to adopt hereafter an engineering notation for the

TABLE I

List of Symbols

a_{ij}	coordinate transformation constants $(i,j = 1,2,3)$
A,B,C,D,F,G,H,J	amplitude (meters)
c	arbitrary constant
$c_{ijkl}(c_{pq})$	elastic stiffness constant at constant electric field (newton/meter^2) $(i,j,k,l = 1,2,3)$ $(p,q = 1,2,3\ldots6)$
f,g,j,q,r,s	arbitrary functions
D_i	component of electric displacement (coulomb/meter^2)
$e_{mij}(e_{mp})$	Piezoelectric constant (newton/volt. meter)
E_m	component of electric field (volt/meter)
h	half thickness of plate (meters)
k_i	component of wave number (1/meter)
$S_{kl}(S_q)$	strain (numeric)
$T_{ij}(T_p)$	stress (newton/meter^2)
t	time (seconds)
$u_k(u,v,w)$	component of displacement (meters)
$x_i(x,y,z)$	cartesian coordinates
ϵ_{im}	dielectric constant (coulomb/volt·meter)
Θ_i	angle to i^{th} coordinate axis
λ	equivalent elastic stiffness (newton/meter^2)
ρ	mass density (kilogram/meter^3)
φ	electric potential (volts)
χ	scalar displacement potential
ψ_i	component of vector displacement potential
ω	angular velocity (1/seconds)

stress, strain, and the elastic and piezoelectric constants, bearing in mind that the abbreviated form of the equations is a result of the symmetry of the stress and strain terms. More specifically, since $T_{ij} = T_{ji}$ and $S_{kl} = S_{lk}$, we find that with the indices i, j, k etc. running from 1 to 3, a more abbreviated but equivalent notation is T_p and S_q where p,q etc. run from 1 to 6 and where T_1 corresponds to T_{11}, T_2 to T_{22}, T_3 to T_{33}, T_4 to T_{23}, T_5 to T_{13}, and T_6 to T_{12}. Similarly S_1 corresponds to S_{11}, S_2 to S_{22}, S_3 to S_{33}, but S_4 corresponds to $2S_{23}$, S_5 to $2S_{13}$, and S_6 to $2S_{12}$, the factors of 2 are necessary to retain the equations (II - 1,2,3) in the same form. The fourth order elastic tensor c_{ijkl} reduces to a second order c_{pq} where in a similar manner $p,q = (1,2,3...6)$, and the third order piezoelectric tensor e_{mij} reduces to a second order tensor e_{mp} where $m = (1,2,3)$ and $p = (1,2,3...6)$. In order to minimize the confusion as to the range over which an index is to be carried, the indices i,j,k,l,m and n will be used to denote a range (1,2,3) and the indices p,q,r,s to denote a range (1,2,3...6).

We can now write the equations of state in the abbreviated form

$$T_p = c_{pq}S_q - e_{mp}E_m \qquad \text{(II - 7)}$$

$$D_i = e_{iq}S_q + \epsilon_{im}E_m \qquad \text{(II - 8)}$$

C. Elastic, Piezoelectric, and Dielectric Matrices

The elastic, piezoelectric, and dielectric constants can most conveniently be written in a matrix form. The matrices for the hexagonal system are listed by Nye[18] as

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} = c_{pq} \qquad c_{66} = \tfrac{1}{2}(c_{11} - c_{12})$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} = e_{mp}$$

$$\begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} = \epsilon_{im}$$

The values of the constants for barium titanate are

$c_{11} = 16.6$ $c_{12} = 7.66$ $c_{13} = 7.75$ $c_{33} = 16.2$ $c_{44} = 4.29$

($\times 10^{10}$ newton/meter^2)

$\epsilon_{11} = 1300\epsilon_0 = 115.1$ $\epsilon_{33} = 1200\epsilon_0 = 106.2$

($\times 10^{-10}$ coulomb/volt.meter)

and,

$e_{15} = 10.4$ $e_{31} = -2.55$ $e_{33} = 15.2$ (newton/volt.meter)

$\rho = 5600$ (kilogram/meter^3).

III. ELASTIC WAVE PROPAGATION

A. Background

Before we consider wave propagation in an infinite plate, there are a number of lesser problems which offer insight into the nature of elastic wave propagation in piezoelectric media. Some of the questions to be answered are whether there exists, as in isotropic media, three roots to the wave equations, one characterized by a purely dilitational mode; and the other two by a purely distortional mode, what are the phase velocities of the roots to the wave equations, and what, if any, is the directional dependence of these velocities, which of these roots can be piezoelectrically excited, and which are purely elastic in nature, and then finally in the context of an infinite plate, what kinds of propagation can exist in a plate with traction free electroded faces and what can we expect for a resonance frequency.

The questions presented here are now to be treated for a hexagonal system and some results illustrated numerically by barium titanate.

One of the advantages of the hexagonal system is that it has a radial symmetry about the z, or poled, axis. We shall make use of this by considering only the plane formed by the z (or x_3) axis and an x (or x_1) axis defined as any arbitrarily specified direction lying in the plane normal to the z axis. Our solutions in the x-z plane will then describe the three dimensional solution in an arbitrary plane containing the z axis. In view of the symmetry in the y (or x_2) direction, we can now ignore terms which are y dependent. As a result, we find, for example, that the non-zero strain terms are S_1, S_3, and S_5.

B. Hexagonal Elastic Media

Before introducing the additional piezoelectric terms, consider first elastic waves in a hexagonal medium. We shall assume only that the x and z displacements (u and w respectively) are circular functions and that they have a time dependence $\exp(-i\omega t)$, i.e.

$$u = A_1 \exp i(k_1 x + k_3 z - \omega t) \qquad \text{(III - 1)}$$

$$w = A_3 \exp i(k_1 x + k_3 z - \omega t) \qquad \text{(III - 2)}$$

The non-zero terms of our wave equation (II - 5), neglecting the piezoelectric terms are

$$\rho u_{,tt} = c_{11} u_{,xx} + c_{44} u_{,zz} + (c_{13} + c_{44}) w_{,xz} \qquad \text{(III - 3)}$$

$$\rho w_{,tt} = (c_{13} + c_{44}) u_{,xz} + c_{44} w_{,xx} + c_{33} w_{,zz} \qquad \text{(III - 4)}$$

Substituting the expressions (III - 1,2) for the displacements and simplifying we have

$$(c_{11} k_1^2 + c_{44} k_3^2 - \rho\omega^2) A_1 + (c_{13} + c_{44}) k_1 k_3 A_3 = 0 \qquad \text{(III - 5)}$$

$$(c_{13} + c_{44}) k_1 k_3 A_1 + (c_{44} k_1^2 + c_{33} k_3^2 - \rho\omega^2) A_3 = 0 \qquad \text{(III - 6)}$$

A non-trivial solution requires the determinant of the coefficients to be zero. This gives a velocity equation,

$$(c_{11} k_1^2 + c_{44} k_3^2 - \rho\omega^2)(c_{44} k_1^2 + c_{33} k_3^2 - \rho\omega^2) = (c_{13} + c_{44})^2 k_1^2 k_3^2 \qquad \text{(III - 7)}$$

which specifies two roots for a given direction. The roots for propagation along the x axis ($k_3 = 0$) and along the z axis ($k_1 = 0$) are immediately available and are: $\left.\frac{\omega}{k}\right]_x = \left[\frac{c_{11}}{\rho}\right]^{1/2}$ and $\left[\frac{c_{44}}{\rho}\right]^{1/2}$; and $\left.\frac{\omega}{k}\right]_z = \left[\frac{c_{33}}{\rho}\right]^{1/2}$ and $\left[\frac{c_{44}}{\rho}\right]^{1/2}$.

We see that the velocity $\frac{\omega}{k}$ is dependent on the direction, as was to be expected. The question arises as to what these velocities represent, and as to whether the use of the displacement potential notation, so useful in isotropic problems, would be useful here as well.

C. Displacement Potential Notation

The displacement potentials χ and $\vec{\psi}$ are defined by

$$u = \chi_{,x} + \psi_{z,y} - \psi_{y,z} \qquad \text{(III - 8)}$$

$$v = \chi_{,y} + \psi_{x,z} - \psi_{z,x} \qquad \text{(III - 9)}$$

$$w = \chi_{,z} + \psi_{y,x} - \psi_{x,y} \qquad \text{(III - 10)}$$

where χ is a scalar and represents the propagation which is entirely dilitational in nature, whereas the vector potential $\vec{\psi}$

represents the propagation which is entirely distortional in nature (shear). Under our hexagonal symmetry, these equations (III - 8,9,10) reduce to

$$u = \chi_{,x} - \psi_{,z} \qquad \text{(III - 11)}$$

$$w = \chi_{,z} + \psi_{,x} \qquad \text{(III - 12)}$$

Substituting these expressions for the displacement into the wave equations (III - 3,4), we have

$$\rho(\chi_{,x} - \psi_{,z})_{,tt} = c_{11}(\chi_{,xxx} - \psi_{,xxz})$$
$$+ c_{13}(\chi_{,xzz} + \psi_{,xxz}) + c_{44}(2\chi_{,xzz} + \psi_{,xxz} - \psi_{,zzz}) \qquad \text{(III - 13)}$$

$$\rho(\chi_{,z} + \psi_{,x})_{,tt} = c_{44}(2\chi_{,xxz} + \psi_{,xxx} - \psi_{,xzz})$$
$$+ c_{13}(\chi_{,xxz} - \psi_{,xzz}) + c_{33}(\chi_{,zzz} + \psi_{,xzz}) \qquad \text{(III - 14)}$$

If we assume a purely dilitational, or curl-free, propagation, we will have all ψ terms zero. Furthermore, an assumption that χ is as well a function of the form,

$$\chi = \chi \exp i(k_1 x + k_3 z - \omega t) \qquad \text{(III - 15)}$$

will, after simplification, reduce the two equations (III - 13,14) to

$$\rho\omega^2 = c_{11} k_1^2 + (c_{13} + 2c_{44})\, k_3^2 \qquad \text{(III - 16)}$$

$$\rho\omega^2 = (c_{13} + 2c_{44}) k_1^2 + c_{33} k_3^2 \qquad \text{(III - 17)}$$

which cannot be simultaneously satisfied except along that cone of revolution offset from the z axis Θ_3 degrees where,

$$\tan \Theta_3 = \left[\frac{c_{33} - c_{13} - 2c_{44}}{c_{11} - c_{13} - 2c_{44}}\right]^{\frac{1}{2}} \qquad \text{(III - 18)}$$

Had this been an isotropic medium, we would have had $c_{11} = c_{33} = \lambda + 2\mu$ and $c_{13} = \lambda$, $c_{44} = \mu$ where λ and μ are Lame's constants, so that the equations (III - 16,17) would have been everywhere satisfied and the dilitational velocity specified as $\frac{\omega}{k} = \left[\frac{c_{11}}{\rho}\right]^{\frac{1}{2}}$.

A similar treatment under the assumption of a purely distortional mode (all χ terms zero) gives the analogous requirements,

$$\rho\omega^2 = (c_{11} - c_{13} - c_{44})k_1^2 + c_{44}k_3^2 \qquad \text{(III - 19)}$$

$$\rho\omega^2 = c_{44}k_1^2 + (c_{33} - c_{13} - c_{44})k_3^2 \qquad \text{(III - 20)}$$

which are as well, simultaneously satisfied only under the condition (III - 18), and which reduce in the isotropic case to the shear velocity $\frac{\omega}{k} = \left[\frac{c_{44}}{\rho}\right]^{\frac{1}{2}}$.

It should be noted that purely distortional and purely dilitational modes are also possible along the crystallographic axes. If we assume the u displacement zero, then the equations (III - 3,4) become in the displacement potential notation

$$0 = (c_{13} + c_{44})(\chi_{,xzz} + \psi_{,xxz}) \qquad \text{(III - 21)}$$

$$-\rho\omega^2(\chi_{,z} + \psi_{,x}) = c_{44}(\chi_{,xxz} + \psi_{,xxx})$$
$$+ c_{33}(\chi_{,zzz} + \psi_{,xzz}) \qquad \text{(III - 22)}$$

If we now select a direction along the x axis, all terms with a z partial derivative become zero, satisfying (III - 21) identically and giving a purely distortional mode from (III - 22), since we now have only ψ terms. Similarly a choice of propagation along the z axis will eliminate all terms with an x partial derivative satisfying (III - 21) identically, and yielding a purely dilitational wave (χ terms only) from (III - 22).

If we had selected the w displacement as zero, then the equations (III - 3,4) would have reduced to

$$-\rho\omega^2(\chi_{,x} - \psi_{,z}) = c_{11}(\chi_{,xxx} - \psi_{,xxz}) + c_{44}(\chi_{,xzz} - \psi_{,zzz}) \qquad \text{(III - 23)}$$

$$0 = (c_{13} + c_{44})(\chi_{,xxz} - \psi_{,xzz}) \qquad \text{(III - 24)}$$

Selection now of the x axis direction will eliminate all terms with a partial derivative with respect to z, satisfying (III - 24) identically and yielding a purely dilitational mode (χ terms only) from (III - 23). A choice of propagation along the z axis will eliminate all x dependent terms, satisfying (III - 24) identically and giving a purely distortional mode (ψ terms only) from (III - 23).

In each of the above cases, the dilitational mode has been associated with the elastic constants c_{11} and c_{33}, whereas the distortional mode has been associated with the constant c_{44}.

We see then that the elastic propagation in a hexagonal system is in general not purely distortional, nor purely dilitational, but rather a predominantly distortional, and a predominantly dilitational mode. We will not be able to make profitable use of the displacement potential notation as a result of the fact that the equations do not separate except along three unique directions. We will rely on identifying the nature of the propagation mode by the association of the elastic constants c_{11} and c_{33}, or their approximate numeric equivalent, with the dilitational modes, and the constant c_{44} or its equivalent with the distortional modes.

D. Hexagonal Piezoelectric Media

In view of the preceding, we will expect elastic waves in a piezoelectric medium which are predominantly either longitudinal (dilitational), or transverse (distortional) and which have a velocity which is directionally dependent. In addition, we will expect some reflection of the piezoelectric effect in the velocities, i.e. some piezoelectric stiffening which could also be directionally dependent.

We have now three equations to be satisfied simultaneously: the two wave equations (II - 5), and the Maxwell equation (II - 6). We should mention at this point that the Maxwell equation (II - 6) which is customarily seen as $\vec{\nabla}\cdot\vec{D}=\rho$ corresponds to an assumption that there are internal to the piezoelectric medium, no free charge sources.

Beginning with the elastic, piezoelectric, and dielectric matrices for a hexagonal medium listed in section II, we find that the non-zero terms of the stress displacement are

$$T_1 = c_{11} u_{,x} + c_{13} w_{,z} + e_{31} \varphi_{,z} \qquad \text{(III - 25)}$$

$$T_3 = c_{13} u_{,x} + c_{33} w_{,z} + e_{33} \varphi_{,z} \qquad \text{(III - 26)}$$

$$T_5 = c_{44} u_{,z} + c_{44} w_{,x} + e_{15} \varphi_{,x} \qquad \text{(III - 27)}$$

$$D_x = e_{15} u_{,z} + e_{15} w_{,x} - \epsilon_{11} \varphi_{,x} \qquad \text{(III - 28)}$$

$$D_z = e_{31} u_{,x} + e_{33} w_{,z} - \epsilon_{33} \varphi_{,z} \qquad \text{(III - 29)}$$

The non-zero terms of the two wave equations, including now the piezoelectric terms are

$$\rho u_{,tt} = c_{11} u_{,xx} + c_{44} u_{,zz} + (c_{13} + c_{44}) w_{,xz} + (e_{15} + e_{31}) \varphi_{,xz} \qquad \text{(III - 30)}$$

$$\rho w_{,tt} = (c_{13} + c_{44}) u_{,xz} + c_{44} w_{,xx} + c_{33} w_{,zz} + e_{15} \varphi_{,xx} + e_{33} \varphi_{,zz} \qquad \text{(III - 31)}$$

and the Maxwell euqation (II - 6) becomes

$$(e_{15} + e_{31}) u_{,xz} + e_{15} w_{,xx} + e_{33} w_{,zz} - \epsilon_{11} \varphi_{,xx} - \epsilon_{33} \varphi_{,zz} = 0 \qquad \text{(III - 32)}$$

Once again assuming displacements of the general form (III - 1,2) and an electric potential of the form

$$\varphi = A_4 \exp i(k_1 x + k_3 z - \omega t) \qquad \text{(III - 33)}$$

we find that substitution of these assumptions into the three equations (III - 30, 31, 32) gives us the three homogeneous equations.

$$(c_{11} k_1^2 + c_{44} k_3^2 - \rho \omega^2) A_1 + k_1 k_3 (c_{13} + c_{44}) A_3 + k_1 k_3 (e_{31} + e_{15}) A_4 = 0 \qquad \text{(III - 34)}$$

$$k_1k_3(c_{13}+c_{44})A_1 + (c_{44}k_1^2 + c_{33}k_3^2-\rho\omega^2)A_3 + (e_{15}k_1^2 + e_{33}k_3^2)A_4 = 0 \quad \text{(III - 35)}$$

$$k_1k_3(e_{15}+e_{31})A_1 + (e_{15}k_1^2 + e_{33}k_3^2)A_3 - (\epsilon_{11}k_1^2 + \epsilon_{33}k_3^2)A_4 = 0 \quad \text{(III - 36)}$$

Once again, a non-trivial solution requires that the determinant of the coefficients of the amplitudes be zero. The two characteristic values of $\rho\omega^2$ that satisfy the dederminant

$$\begin{vmatrix} (c_{11}k_1^2 + c_{44}k_3^2-\rho\omega^2) & k_1k_3(c_{13}+c_{44}) & k_1k_3(e_{31}+e_{15}) \\ k_1k_3(c_{13}+c_{44}) & (c_{44}k_1^2+c_{33}k_3^2-\rho\omega^2) & (e_{15}k_1^2+e_{33}k_3^2) \\ k_1k_3(e_{31}+e_{15}) & (e_{15}k_1^2+e_{33}k_3^2) & -(\epsilon_{11}k_1^2+\epsilon_{33}k_3^2) \end{vmatrix} = 0$$

(III - 37)

will specify the phase velocity for a given direction.

Once again there are two directions which readily give a root, namely propagation along the x axis ($k_3 = 0$) and along the z axis ($k_1 = 0$). The two roots for these cases are $\left.\frac{\omega}{k}\right|_x = \left[\frac{c_{11}}{\rho}\right]^{\frac{1}{2}}$ and $\left[\frac{c_{44}}{\rho}+\frac{e_{15}^2}{\epsilon_{11}\rho}\right]^{\frac{1}{2}}$; and $\left.\frac{\omega}{k}\right|_z = \left[\frac{c_{33}}{\rho}+\frac{e_{33}^2}{\epsilon_{33}\rho}\right]^{\frac{1}{2}}$, and $\left[\frac{c_{44}}{\rho}\right]^{\frac{1}{2}}$.

Numerically these velocities for barium titanate are

$\left.\frac{\omega}{k}\right|_x = 5.44 \times 10^3$ and 3.06×10^3 (meters/second), and

$\left.\frac{\omega}{k}\right|_z = 5.72 \times 10^3$ and 2.77×10^3 (meters/second).

We see then that along the z axis there is a piezoelectrically stiffened longitudinal wave and a purely elastic shear whereas along the x axis there is possible a purely elastic longitudinal wave and a piezoelectrically stiffened shear. It is evident from the form of the determinant (III - 37) that for directions other than along either axis, both the longitudinal and the shear modes are going to be piezoelectrically stiffened, and therefore, conversely, both longitudinal and shear modes are likely to be generated piezoelectrically when we are concerned with directions of wave propagation other than along either of the axes.

Other roots of the determinant (III - 37) for intermediate directions between the x and z axes for barium titanate are listed in Table II below.

E. The Restricted Case $\varphi = \varphi(z)$

A number of possibilities exist now as to the direction to proceed. Before we consider the problem of the infinite plate, the assumption which is to be then made that the electric potential φ is a function of the z coordinate only might profitably be considered in the above case. The immediate results are not dramatic, however, the equations (III - 30,31,32) have been simplified somewhat and the determinant (III - 37) has now been considerably simplified by the presence of a zero term, i.e. we now have

$$\begin{vmatrix} (c_{11}k_1^2 + c_{44}k_3^2 - \rho\omega^2) & k_1k_3(c_{13}+c_{44}) & 0 \\ k_1k_3(c_{13}+c_{44}) & (c_{44}k_1^2 + c_{33}k_3^2 - \rho\omega^2) & e_{33}k_3^2 \\ k_1k_3(e_{15}+e_{31}) & (e_{15}k_1^2 + e_{33}k_3^2) & -\epsilon_{33}k_3^2 \end{vmatrix} = 0 \qquad \text{(III - 38)}$$

the expansion of which is

TABLE II

Velocity Profile of Barium Titanate

Angle from z Axis (θ_3)	Longitudinal	Shear	Shear*
0°	5.72	2.77	2.77
10°	5.72	2.77	2.77
20°	5.70	2.79	2.78
30°	5.68	2.82	2.78
40°	5.62	2.88	2.79
45°	5.61	2.89	2.79
50°	5.61	2.90	2.80
55°	5.58	2.92	2.80
60°	5.53	2.97	2.81
70°	5.49	3.01	2.82
80°	5.47	3.04	2.82
90°	5.44	3.06	2.82

all velocities are x 10^3 meter/second.

*A purely elastic y-dependent mode introduced in Section III-F.

$$(c_{11}k_1^2 + c_{44}k_3^2 - \rho\omega^2)\left[(c_{44} + \frac{e_{15}e_{33}}{\epsilon_{33}})k_1^2 + (c_{33} + \frac{e_{33}^2}{\epsilon_{33}})k_3^2 - \rho\omega^2\right]$$
$$= k_1^2k_3^2\left[(c_{13} + c_{44})^2 + \frac{e_{33}}{\epsilon_{33}}(e_{15} + e_{31})(c_{13} + c_{44})\right] \quad (III - 39)$$

Comparison of this with the equation (III - 7) will show that the two equations are similar, the latter being of the same form with the addition of piezoelectric terms.

F. "Y" Dependent Modes

One other question arises at this point as to what has become of the third root to the wave equations which we know to exist in an isotropic medium. The answer is that we implicitly ignored this root, which is a horizontally (x-y plane) polarized shear wave, when we assumed no y (or x_2) dependence due to the crystal symmetry. In order to justify the assumption, we find that by admitting all terms with y dependence, and as well assuming that all three displacements (u, v, and w) are y dependent and then proceeding in an entirely analogous manner as above, we find that there are now three wave equations plus the Maxwell equation, which are after simplification

$$(c_{11}k_1^2 + c_{66}k_2^2 + c_{44}k_3^2 - \rho\omega^2)A_1 + k_1k_2(c_{12} + c_{66})A_2$$
$$+ k_1k_3(c_{13} + c_{44})A_3 + k_1k_3(e_{31} + e_{15})A_4 = 0 \quad (III - 40)$$

$$k_1k_2(c_{66} + c_{12})A_1 + (c_{66}k_1^2 + c_{11}k_2^2 + c_{44}k_3^2 - \rho\omega^2)A_2$$
$$+ k_2k_3(c_{13} + c_{44})A_3 + k_2k_3(e_{31} + e_{15})A_4 = 0 \quad (III - 41)$$

$$k_1k_3(c_{13} + c_{44})A_1 + k_2k_3(c_{13} + c_{44})A_2 + (c_{44}k_1^2 + c_{44}k_2^2 + c_{33}k_3^2 - \rho\omega^2)A_3$$
$$+ (e_{15}k_1^2 + e_{15}k_2^2 + e_{33}k_3^2)A_4 = 0 \quad (III - 42)$$

$$k_1k_3(e_{15}+e_{31})A_1+k_2k_3(e_{15}+e_{31})A_2+(e_{15}k_1^2+e_{15}k_2^2+e_{33}k_3^2)A_3$$
$$-(\epsilon_{11}k_1^2+\epsilon_{11}k_2^2+\epsilon_{33}k_3^2)A_4=0 \qquad \text{(III - 43)}$$

A non-trivial solution will require the determinant of the coefficients to be zero, however, we can see that if we are to consider propagation only in the (x-z) plane ($k_2 = 0$) then the entire equation (III - 41) reduces to its second term (which is also decreased one term), viz.

$$0A_1+(c_{66}k_1^2+c_{44}k_3^2-\rho\omega^2)A_2+0A_3+0A_4 = 0 \qquad \text{(III - 41)}'$$

Furthermore, under these conditions each of the second terms in the remaining equations (III - 40, 41, 42) are zero, simplifying the determinant of the coefficients to the form

$$\begin{vmatrix} (c_{11}k_1^2+c_{44}k_3^2-\rho\omega^2) & 0 & k_1k_3(c_{13}+c_{44}) & k_1k_3(e_{31}+e_{15}) \\ 0 & (c_{66}k_1^2+c_{44}k_3^2-\rho\omega^2) & 0 & 0 \\ k_1k_3(c_{13}+c_{44}) & 0 & (c_{44}k_1^2+c_{33}k_3^2-\rho\omega^2) & (e_{15}k_1^2+e_{33}k_3^2) \\ k_1k_3(e_{15}+e_{31}) & 0 & (e_{15}k_1^2+e_{33}k_3^2) & -(\epsilon_{11}k_1^2+\epsilon_{33}k_3^2) \end{vmatrix} = 0$$

(III - 44)

It can be readily seen that determinant (III - 44) is equivalent to the term $(c_{66}k_1^2+c_{44}k_3^2-\rho\omega^2)$ multiplying the determinant (III - 37), which we have previously considered. The conclusion then is that our third permissible mode is specified by the satisfaction of

$$(c_{66}k_1^2 + c_{44}k_3^2-\rho\omega^2) = 0 \qquad \text{(III - 45)}$$

and represents a purely elastic shear mode associated with a y (or x_2) displacement.

It is interesting to note that had we chosen a direction of propagation in the (y-z) plane ($k_1 = 0$) the determinant of the coefficients of (III - 40,41,42,43) would become

$$\begin{vmatrix} (c_{66}k_2^2 + c_{44}k_3^2 - \rho\omega^2) & 0 & 0 & 0 \\ 0 & (c_{11}k_2^2 + c_{44}k_3^2 - \rho\omega^2) & k_2k_3(c_{13} + c_{44}) & k_2k_3(e_{31} + e_{15}) \\ 0 & k_2k_3(c_{13} + c_{44}) & (c_{44}k_2^2 + c_{33}k_3^2 - \rho\omega^2) & (e_{15}k_2^2 + e_{33}k_3^2) \\ 0 & k_2k_3(e_{31} + e_{15}) & (e_{15}k_2^2 + e_{33}k_3^2) & -(\epsilon_{11}k_2^2 + \epsilon_{33}k_3^2) \end{vmatrix} = 0 \qquad \text{(III - 46)}$$

This can be seen to be entirely equivalent to (III - 44) provided k_2 replaces k_1, confirming the z axis symmetry.

G. <u>A Piezoelectric Infinite Plate</u>

Consider now an infinite piezoelectric plate, coated with massless conducting electrodes on each of two faces normal to the z axis, and separated by a thickness $2h$. We impress a sinusoidal voltage on the electrodes, and will accordingly specify the boundary condition

$$\varphi(\pm h) = \pm \varphi_0 \exp(-i\omega t) \qquad \text{(III - 47)}$$

The assumption to be made at this point is that the potential field within the medium is a function of the z coordinate only. We justify this assumption on the grounds that we are dealing with an infinite plate and can ignore edge effects and secondly, that in most ferroelectric ceramics the dielectric constant is extremely high.

It will be profitable now to investigate the implications of a z dependent electric potential. Let the displacements u and w, and the potential φ be specified by

$$u = f(k_1 x + k_3 z)\ \exp(-i\omega t) \qquad \text{(III - 48)}$$

$$w = g(k_1 x + k_3 z)\ \exp(-i\omega t) \qquad \text{(III - 49)}$$

$$\varphi = j(z)\ \exp(-i\omega t) \qquad \text{(III - 50)}$$

where the functions f, g, and j are completely arbitrary, as are the constants k_1 and k_3. Substitution of the equations (III - 48,49,50) into the wave equations (III - 26,27) and Maxwell's equation (III - 28) gives us

$$(c_{11} k_1^2 + c_{44} k_3^2)\ddot{f} + (c_{13} + c_{44})k_1 k_3 \ddot{g} + 0\ \ddot{j} = \rho\omega^2 f \qquad \text{(III - 51)}$$

$$(c_{13} + c_{44})k_1 k_3 \ddot{f} + (c_{44} k_1^2 + c_{33} k_3^2)\ddot{g} + e_{33}\ddot{j} = \rho\omega^2 g \qquad \text{(III - 52)}$$

$$(e_{15} + e_{31})k_1 k_3 \ddot{f} + (e_{15} k_1^2 + e_{33} k_3^2)\ddot{g} - \epsilon_{33}\ddot{j} = \quad 0 \qquad \text{(III - 53)}$$

where $\dfrac{d^2 f(k_1 x + k_3 z)}{d(k_1 x + k_3 z)^2}$ has been designated $\ddot{f}$ etc. We now have a series of non-homogeneous equations, which for a given circular frequency ω specify the three variables f, g, and j in terms of a linear combination of the functions f and g. Assuming only that the determinant of the coefficients of the primed variables is not zero, we could obtain by Cramer's rule or other means an expression

$$\ddot{j}(z) = c_1 f(k_1 x + k_3 z) + c_2 g(k_1 x + k_3 z) \qquad \text{(III - 54)}$$

where c_1 and c_2 are constants dependent only on the constants of equations (III - 51,52,53).

In the choice of form of equations (III - 48,49,50), we have implicitly restricted ourselves to those particular solutions of the wave equation which are exponential functions, the circular and hyperbolic functions. These are the solutions which are physically significant. Bearing in mind that we are now concerned with exponential forms, we find that in order to satisfy (III - 54) for all values of x, it is necessary that either a.) k_1 be zero giving f and g as z dependent only or b.) $c_1 f$ and $c_2 g$ be so related that their sum is a constant for all values of x, but then their sum will be a constant for all values of z as well, to the end that the left-hand side $\vec{j}(z)$ is a constant and not an exponential function of z as predicated. The result then is that the only particular solutions of physical significance are those in which all three functions f, g, and j are z dependent only.

The conclusion that the displacements as well as the electric potential are z-dependent only will seem more credible if we consider that in an infinite plate there are no standing wave patterns other than between the two faces, and that any wave not a member of a standing wave pattern and propagating in a direction other than along the z axis, will generate an accompanying electric potential which is as well x dependent, violating our initial assumption. An apparent exception to this is the purely elastic wave which we have seen can exist along the x axis, however, this particular wave has no piezoelectric terms and accordingly makes no contribution to the electric field.

Accepting a z-only dependence for the displacements and the electric potential, we find that the equations (III - 30,31,32) reduce now to

$$\rho u_{,tt} = c_{44} u_{,zz} \qquad \text{(III - 55)}$$

$$\rho w_{,tt} = c_{33} w_{,zz} + e_{33} \varphi_{,zz} \qquad \text{(III - 56)}$$

$$0 = e_{33} w_{,zz} - \epsilon_{33} \varphi_{,zz} \qquad \text{(III - 57)}$$

Since only the anti-symmetric modes are piezoelectrically generated, we will want now to specify circular functions for the displacements u and w of the form

$$u = A \sin kz \exp(-i\omega t) \qquad \text{(III - 58)}$$

$$w = C \sin kz \exp(-i\omega t) \qquad \text{(III - 59)}$$

Substituting (III - 59) into (III - 57) and integrating both sides twice with respect to z gives

$$\varphi = \frac{Ce_{33}}{\epsilon_{33}} \sin kz + c_1 z + c_2 \qquad \text{(III - 60)}$$

where we have neglected the $\exp(-i\omega t)$ time dependence common to all terms (as will be the practice throughout the remainder of this paper). The two constants c_1 and c_2 of (III - 60) can be evaluated by applying the boundary condition

(III - 47). Having done this, we then have

$$\varphi = \frac{Ce_{33}}{\epsilon_{33}}\sin kz + \left[\frac{\varphi_0}{h} - \frac{Ce_{33}}{\epsilon_{33}}\frac{\sin kh}{h}\right] z \qquad \text{(III - 61)}$$

Consider now the additional boundary conditions that

$$T_5\ (\pm\ h) = 0 \qquad \text{(III - 62)}$$

$$T_3\ (\pm\ h) = 0 \qquad \text{(III - 63)}$$

which require the two plate faces ($z = \pm h$) to be traction free.

We see from (III - 23) that the stress T_5 on $z = \pm h$ reduces to

$$T_5(\pm h) = C_{44}u_{,z}\,(\pm h) = 0 \qquad \text{(III - 64)}$$

and from (III - 26) that the stress T_3 on $z = \pm h$ becomes

$$T_3(\pm h) = C_{33}W_{,z}\,(\pm h) + e_{33}\varphi_{,z}(\pm h) = 0 \qquad \text{(III - 65)}$$

Substituting (III - 58,59,61) into (III - 64,65), we have

$$C_{44}Ak\ \cos kh = 0 \qquad \text{(III - 66)}$$

$$\left[\left(C_{33} + \frac{e_{33}^2}{\epsilon_{33}}\right)kh\cos kh - \frac{e_{33}}{\epsilon_{33}}\frac{\sin kh}{h}\right]C + \varphi_0 e_{33} = 0 \qquad \text{(III - 67)}$$

Equation (III - 67) specifies the amplitude C as a function of various constants and the wave number k. We will expect resonance for that value of k which brings the bracketed coefficient of C to zero, demanding an infinite amplitude C to satisfy the equation. It is easily seen that the values of k which accomplish this are those which are the roots to the relation

$$\tan kh = kh\left(1 + \frac{C_{33}\epsilon_{33}}{e_{33}^2}\right) \qquad \text{(III - 68)}$$

The resonance frequencies of an infinite plate will therefore not be integral multiples of the first resonant frequency but rather the roots to (III - 68) which are (most materials have a fairly large value for the factor $\frac{C_{33}\epsilon_{33}}{e_{33}^2}$) kh equal to $\pi/2$ minus a small amount, then $3\pi/2$ minus a smaller amount, and so on, approaching for higher frequencies multiples of $(2n-1)\pi/2$. As an example, the first three roots for barium titanate ($\frac{C_{33}\epsilon_{33}}{e_{33}^2} = 7.44$) are $kh)_1 = 1.49$ (85.4°) $kh)_2 = 4.69$ (268.6°), and $kh)_3 = 7.84$(449.1°).

Satisfaction of the equation (III - 66) will require that either $kh = (2n-1)\pi/2$ $(n = 1,2,...)$, or A equal zero. This equation corresponds to the requirement of a stress-free face relative to the shear mode (III - 58). This mode is not excited piezoelectrically, so we may dismiss it with the statement that if it were excited mechanically, it would resonate for $kh = (2n-1)\pi/2$.

We have seen that in an infinite plate, the two displacements u and w must be z-only dependent as well as the potential field; furthermore, a sinusoidal driving potential generates only a thickness longitudinal mode which is resonant at frequencies which are not quite multiples of the first. We have also seen that for directions other than along the z and x axes both roots to the wave equations contain piezoelectric terms so that we would expect both a longitudinal and a shear to be generated for these intermediate directions. We would expect that in an infinite plate which is composed of a material for which the z, or poled, axis does not correspond with the normal to the plate faces, that there would be generated piezoelectrically both a longitudinal mode and a shear, and that resonance would be specified by the satisfaction of a free face in the presence of both. We investigate this case now.

IV. ROTATION ABOUT THE Y AXIS (a "rotated" plate)

A. Transformation of the Elastic, Piezoelectric, and Dielectric Constants

We want now to consider a transformation of coordinates which consists of a simple angular rotation about the y (or x_2) axis resulting in a displacement of the x and z axes an angle Θ degrees from their original directions. All quantities which are with respect to the new rotated coordinates will be designated by a prime. We are concerned with a transformation between two cartesian coordinate systems and need not concern ourselves with the aspects of co-variance and contra-variance. All of our equations of state, the wave equations, and Maxwell's equation will remain form invariant; however, since we are now selecting a z axis which is other than the crystallographic, or poled, axis, we will produce elastic, piezoelectric, and dielectric matrices which are more complex, since we are now longer in the principle axis system.

The coordinates then are to be transformed as

$$x'_i = a_{ij} x_j \qquad \text{(IV - 1)}$$

where the constants a_{ij} correspond to

$$\begin{aligned} &a_{11} = a_{33} = \cos\Theta \quad a_{13} = -a_{31} = \sin\Theta \\ &a_{22} = 1 \qquad a_{12} = a_{21} = a_{23} = a_{32} = 0 \end{aligned} \qquad \text{(IV - 2)}$$

The first task will now be the transformation of each of the fourth order elastic constants to its equivalent in the primed system under the transform

$$c'_{mnop} = a_{mi} a_{nj} a_{ok} a_{pl} c_{ijkl} \qquad \text{(IV - 3)}$$

There are fortunately many zero terms so that these transforms are not as lengthy as they might at first seem. The first few constants are found, for example, to be

$$c'_{11} = (a_{11})^4 c_{11} + (a_{11} a_{13})^2 (2c_{13} + 4c_{55}) + (a_{13})^4 c_{33} \qquad \text{(IV - 4)}$$

$$c'_{12} = (a_{11})^2 c_{12} + (a_{13})^2 c_{13} \qquad \text{(IV - 5)}$$

$$c'_{13} = (a_{11})^4 c_{13} + (a_{11} a_{13})^2 (c_{11} + c_{33} - 4c_{55}) + (a_{13})^4 c_{13} \qquad \text{(IV - 6)}$$

Similarly it is necessary to transform the piezoelectric constants

$$e'_{mno} = a_{mi} a_{nj} a_{ok} e_{ijk} \qquad \text{(IV - 7)}$$

and the dielectric constants

$$\epsilon'_{mn} = a_{mi} a_{nj} \epsilon_{ij} \qquad \text{(IV - 8)}$$

We find as a result that, for example

$$e'_{11} = 2(a^2_{11} a_{13}) e_{15} + (a^2_{11} a_{13}) e_{31} + (a^3_{13}) e_{33} \qquad \text{(IV - 9)}$$

and

$$\epsilon'_{11} = (a^2_{11}) \epsilon_{11} + (a^2_{13}) \epsilon_{33} \qquad \text{(IV - 10)}$$

Continuing with this process for each of the constants gives elastic, piezoelectric and dielectric matrices which are of the form

$$\begin{bmatrix} c'_{11} & c'_{12} & c'_{13} & 0 & c'_{15} & 0 \\ c'_{21} & c'_{22} & c'_{23} & 0 & c'_{25} & 0 \\ c'_{13} & c'_{23} & c'_{33} & 0 & c'_{35} & 0 \\ 0 & 0 & 0 & c'_{44} & 0 & c'_{46} \\ c'_{15} & c'_{25} & c'_{35} & 0 & c'_{55} & 0 \\ 0 & 0 & 0 & c'_{46} & 0 & c'_{66} \end{bmatrix} = c'_{pq} \qquad \text{(IV - 11)}$$

$$\begin{bmatrix} e'_{11} & e'_{12} & e'_{13} & 0 & e'_{15} & 0 \\ 0 & 0 & 0 & e'_{24} & 0 & e'_{26} \\ e'_{31} & e'_{32} & e'_{33} & 0 & e'_{35} & 0 \end{bmatrix} = e'_{mp} \qquad \text{(IV - 12)}$$

$$\begin{bmatrix} \epsilon'_{11} & 0 & \epsilon'_{13} \\ 0 & \epsilon'_{22} & 0 \\ \epsilon'_{13} & 0 & \epsilon'_{33} \end{bmatrix} = \epsilon'_{ij} \qquad \text{(IV - 13)}$$

B. The Restricted Case $\varphi = \varphi(z')$

The new matrices of equations (IV - 11,12,13) now specify stresses and electric displacements of the form

$$T'_1 = c'_{11} S'_1 + c'_{12} S'_2 + c'_{13} S'_3 + c'_{15} S'_5 + e'_{11} \varphi'_{,x} + e'_{13} \varphi'_{,z} \qquad \text{(IV - 14)}$$

$$T'_3 = c'_{13} S'_1 + c'_{23} S'_2 + c'_{33} S'_3 + c'_{35} S'_5 + e'_{13} \varphi'_{,x} + e'_{33} \varphi'_{,z} \qquad \text{(IV - 15)}$$

$$T_5 = c'_{15} S'_1 + c'_{25} S'_2 + c'_{35} S'_3 + c'_{55} S'_5 + e'_{15} \varphi'_{,x} + e'_{35} \varphi'_{,z} \qquad \text{(IV - 16)}$$

$$D'_x = e'_{11}S'_1 + e'_{12}S'_2 + e'_{13}S'_3 + e'_{15}S'_5$$
$$-\epsilon'_{11}\varphi'_{,x} - \epsilon'_{13}\varphi'_{,z} \qquad \text{(IV - 17)}$$
$$D'_z = e'_{31}S'_1 + e'_{32}S'_2 + e'_{33}S'_3 + e'_{35}S'_5$$
$$-\epsilon'_{13}\varphi'_{,x} - \epsilon'_{33}\varphi'_{,z} \qquad \text{(IV - 18)}$$

which after we neglect all y coordinate dependence become

$$T'_1 = c'_{11}u'_{,x} + c_{15}u'_{,z} + c'_{15}w'_{,x} + c'_{13}w'_{,z}$$
$$+e'_{11}\varphi'_{,x} + e'_{31}\varphi'_{,z} \qquad \text{(IV - 19)}$$
$$T'_3 = c'_{13}u'_{,x} + c'_{35}u'_{,z} + c'_{35}w'_{,x} + c'_{33}w'_{,z}$$
$$+e'_{13}\varphi'_{,x} + e'_{33}\varphi'_{,z} \qquad \text{(IV - 20)}$$
$$T'_5 = c'_{15}u'_{,x} + c'_{55}u'_{,z} + c'_{55}w'_{,x} + c'_{35}w'_{,z}$$
$$+e'_{15}\varphi'_{,x} + e'_{35}\varphi'_{,z} \qquad \text{(IV - 21)}$$
$$D'_{x,x} = e'_{11}u'_{,xx} + e'_{15}u'_{,xz} + e'_{15}w'_{,xx} + e'_{13}w'_{,xz}$$
$$-\epsilon'_{11}\varphi'_{,xx} - \epsilon'_{13}\varphi'_{,xz} \qquad \text{(IV - 22)}$$
$$D'_{z,z} = e'_{31}u'_{,xz} + e'_{35}u'_{,zz} + e'_{35}w'_{,xz} + e'_{33}w'_{,zz}$$
$$-\epsilon'_{13}\varphi_{,xz} - \epsilon'_{33}\varphi'_{,zz} \qquad \text{(IV - 23)}$$

The wave equations (II - 5) are unchanged in form. They now have the following non-zero terms in the rotated system.

$$\rho u'_{,tt} = c'_{11}u'_{,xx} + 2c'_{15}u'_{,xz} + c'_{55}u'_{,zz} + c'_{15}w'_{,xx}$$
$$(c'_{13} + c'_{55})w'_{,xz} + c'_{35}w'_{,zz} + e'_{11}\varphi'_{,xx} + (e'_{15} + e'_{31})\varphi'_{,xz}$$
$$+e'_{35}\varphi'_{,zz} \qquad \text{(IV - 24)}$$

$$\rho w'_{,tt} = c'_{15}u'_{,xx} + (c'_{13} + c'_{55})u'_{,xz} + c'_{35}u'_{,zz} + c'_{55}w'_{,xx}$$
$$+2c'_{35}w'_{,xz} + c'_{33}w'_{,zz} + e'_{15}\varphi'_{,xx} + (e'_{13} + e'_{35})\varphi'_{,xz}$$
$$+e'_{33}\varphi'_{,zz} \qquad \text{(IV - 25)}$$

The Maxwell equation is the sum of the terms of equations (IV - 22,23), viz.

$$e'_{11}u'_{,xx}+(e'_{15}+e'_{31})u'_{,xz}+e'_{35}u'_{,zz}+e'_{15}w'_{,xx}+(e'_{13}+e'_{35})w'_{,xz}$$
$$+e'_{33}w'_{,zz}-\epsilon'_{11}\varphi'_{,xx}-2\epsilon'_{13}\varphi'_{,xz}-\epsilon'_{33}\varphi'_{,zz}=0 \qquad \text{(IV - 26)}$$

It can be seen that we are now in much the same position as with the unrotated plate with the exception of having a number of extra terms. The question to be considered at this point is whether the requirement that the electric field be a function of the z' coordinate only will similarly restrict the u and w displacements to z' dependence only as it did for the unrotated plate. We are fortunate in that we are concerned once again with satisfying simultaneously three equations (IV - 24,25,26) all terms of which are of second order. As a result, we can use the procedure of section III - C to establish the z' dependence of the u and w displacements. More specifically, once again assume a z'-only dependence of the electric potential, and arbitrary exponential functions of x' and z' for the displacements

$$u' = q(k_1x'+k_3z')\exp(-i\omega t) \qquad \text{(IV - 27)}$$
$$w' = r(k_1x'+k_3z')\exp(-i\omega t) \qquad \text{(IV - 28)}$$
$$\varphi' = S(z')\exp(-i\omega t) \qquad \text{(IV - 29)}$$

Substitution of these into the equations (IV - 24,25, 26) gives

$$-\rho\omega^2 q = (c'_{11}k_1^2 + 2c'_{15}k_1k_3 + c'_{55}k_3^2)\ddot{q}$$

$$c'_{15}k_1^2 + (c'_{13} + c'_{55})k_1k_3 + c'_{35}k_3^2 \;\ddot{r} + e'_{35}\ddot{S} \qquad \text{(IV - 30)}$$

$$-\rho\omega^2 r = C'_{15}k_1^2 + (c'_{13} + c'_{55})k_1k_3 + c'_{35}k_3^2 \;\ddot{q}$$

$$(c'_{55}k_1^2 + 2c'_{35}k_1k_3 + c'_{33}k_3^2)\ddot{r} + e'_{33}\ddot{S} \qquad \text{(IV - 31)}$$

$$0 = e'_{11}k_1^2 + (e'_{15} + e'_{31})k_1k_3 + e'_{35}k_3^2 \;\ddot{q}$$

$$e'_{15}k_1^2 + (e'_{15} + e'_{35})k_1k_3 + e'_{33}k_3^2 \;\ddot{r} - \epsilon_{33}\ddot{S} \qquad \text{(IV - 32)}$$

where we have once again designated $\dfrac{d^2q(k_1x'+k_3z')}{d(k_1x'+k_3z')^2}$ by $\ddot{q}$.

The equations (IV - 30,31,32) are a series of non-homogeneous equations completely analogous to the equations (III - 51,52,53), so that again, as long as the determinant of the coefficients of the twice differentiated terms is not zero, we will be able to express $\ddot{S}$ as a linear combination of the arbitrary exponential functions q and r, i.e.

$$\ddot{S} = c_1 q(k_1x' + k_3z') + c_2 r(k_1x'+k_3z') \qquad \text{(IV - 33)}$$

so that we must again specify q and r to be z'-only dependent if we are to require S to be a twice differentiable non-zero exponential function of z'.

C. <u>The Rotated Infinite Plate</u>

Assuming then a z'-only dependence for u', w', and φ' we find the equations (IV - 24,25,26) reduce to

$$\rho u_{,tt} = c'_{55}u'_{,zz} + c'_{35}w'_{,zz} + e'_{35}\varphi'_{,zz} \qquad \text{(IV - 34)}$$

$$\rho w_{,tt} = c'_{35}u'_{,zz} + c'_{33}w'_{,zz} + e'_{33}\varphi'_{,zz} \qquad \text{(IV - 35)}$$

$$0 = e'_{35}u'_{,zz} + e'_{33}w'_{,zz} - \epsilon_{33}\varphi'_{,zz} \qquad \text{(IV - 36)}$$

The anti-symmetric modes are piezoelectrically generated, so we will specify

$$u' = B\sin kz' \exp(-i\omega t) \qquad \text{(IV - 37)}$$

$$w' = D\sin kz' \exp(-i\omega t) \qquad \text{(IV - 38)}$$

After integrating (IV - 36) twice with respect to z' and substituting (IV - 37,38), we have

$$\varphi' = \frac{e'_{35}}{\epsilon'_{33}} B\sin kz' + \frac{e'_{33}}{\epsilon'_{33}} D\sin kz' + c_1 z' + c_2 \qquad \text{(IV - 39)}$$

Application of the boundary condition

$$\varphi(\pm h) = \pm \varphi_0 \exp(-i\omega t) \qquad \text{(IV - 40)}$$

to the equation (IV - 39) will specify the constants c_1 and c_2 to the end that

$$\varphi' = \frac{e'_{35}}{\epsilon'_{33}}\left(\sin kz' - \frac{\sin kh}{h} z'\right)B + \frac{e'_{33}}{\epsilon'_{33}}\left(\sin kz' - \frac{\sin kh}{h} z'\right)D + \frac{\varphi_0}{h} z' \qquad \text{(IV - 41)}$$

and

$$\varphi'_{,zz} = -\frac{e'_{35}k^2}{\epsilon'_{33}} \sin kz' B - \frac{e'_{33}k^2}{\epsilon'_{33}} \sin kz' D \qquad \text{(IV - 42)}$$

Substituting (IV - 42) and (IV - 37,38) into the equations (IV - 34,35) gives us

$$\left[c'_{55}k^2 + \frac{(e'_{35})^2}{\epsilon'_{33}} k^2 - \rho\omega^2\right]B + \left(c'_{35} + \frac{e'_{35}e_{33}}{\epsilon'_{33}}\right)k^2 D = 0 \qquad \text{(IV - 43)}$$

$$\left(c'_{35} + \frac{e_{35}e_{33}}{\epsilon'_{33}}\right)k^2 B + \left[c_{33}k^2 + \frac{(e'_{33})^2}{\epsilon'_{33}} k^2 - \rho\omega^2\right]D = 0 \qquad \text{(IV - 44)}$$

Equations (IV - 43,44) require the determinant of the coefficients zero for a non-trivial solution which specifies our roots to the wave equations as the roots to

$$(c'^{*}_{55} - \lambda)(c'^{*}_{33} - \lambda) = (c'^{*}_{35})^2 \qquad \text{(IV - 45)}$$

where we have designated $c'^{*}_{55} = c'_{55} + \frac{(e'_{35})^2}{\epsilon'_{33}}$, $c'^{*}_{33} = c'_{33} + \frac{(e'_{33})^2}{\epsilon'_{33}}$, $c'^{*}_{35} = c'_{35} + \frac{e_{33} e_{35}}{\epsilon_{33}}$, and $\lambda = \frac{\rho\omega^2}{k^2}$. In view of the importance of the transformation of the constants involved in (IV - 45), they are listed here in full.

$$c'_{33} = (a_{11})^4 c_{33} + (2c_{13} + 4c_{44})(a_{11} a_{13})^2 + (a_{13})^4 c_{11} \qquad \text{(IV - 46)}$$

$$c'_{35} = a_{11} a_{13}^3 (c_{13} + 2c_{44} - c_{11}) + a_{11}^3 a_{13}(c_{33} - c_{13} - 2c_{44}) \qquad \text{(IV - 47)}$$

$$c'_{55} = (a_{11})^4 c_{44} + (a_{11} a_{13})^2 (c_{11} + c_{33} - 2c_{13} - 2c_{44}) + (a_{13})^4 c_{44} \qquad \text{(IV - 48)}$$

$$e'_{33} = (a_{11})^3 e_{33} + a_{11} a_{13}^2 (e_{31} + 2e_{15}) \qquad \text{(IV - 49)}$$

$$e'_{35} = a_{11}^2 a_{13}(e_{33} - e_{15} - e_{31}) + a_{13}^3 e_{15} \qquad \text{(IV - 50)}$$

$$\epsilon'_{33} = a_{11}^2 \epsilon_{33} + a_{13}^2 \epsilon_{11} \qquad \text{(IV - 51)}$$

Equation (IV - 45) reduces at a zero angle of rotation ($a_{11} = 1$, $a_{13} = 0$) to

$$(c_{44} - \lambda)\left[c_{33} + \frac{e_{33}^2}{\epsilon_{33}} - \lambda\right] = 0 \qquad \text{(IV - 52)}$$

and gives the same roots as previously, namely $\frac{\rho\omega^2}{k^2} = c_{44}$ and $\frac{\rho\omega^2}{k^2} = c_{33} + \frac{e_{33}^2}{\epsilon_{33}}$, an elastic shear and a piezoelectrically stiffened longitudinal mode. Similarly, at $\theta = 90^\circ$ we have

$$\left[c_{44} + \frac{(e_{15})^2}{\epsilon_{11}} - \lambda\right](c_{11} - \lambda) = 0 \qquad \text{(IV - 53)}$$

as was also previously established.

We are in a position now to consider satisfaction of the boundary conditions

$$T_3'(\pm h) = T_5'(\pm h) = 0 \tag{IV - 54}$$

for the rotated plate and thereby establish the resonance of the rotated plate.

The stresses T_3' and T_5' reduce for z'-only dependence to

$$T_3' = c_{35}' u'_{,z} + c_{33}' w'_{,z} + e_{33}' \phi'_{,z} \tag{IV - 55}$$

$$T_5' = c_{55}' u'_{,z} + c_{35}' w'_{,z} + e_{35}' \phi'_{,z} \tag{IV - 56}$$

Substitution of (IV - 37,38,41) into the equations (IV - 55,56) evaluated under the conditions (IV - 54) gives

$$T_3'(\pm h) = \left(c_{35}' + \frac{e_{33}' e_{35}'}{\epsilon_{33}'}\right) k\cosh kh - \frac{e_{33}' e_{35}'}{\epsilon_{33}'} \frac{\sinh kh}{h} \quad B$$
$$\left(c_{33}' + \frac{e_{33}'^2}{\epsilon_{33}'}\right) k\cosh kh - \frac{e_{33}'^2}{\epsilon_{33}'} \frac{\sinh kh}{h} \quad D + \frac{\phi_0 e_{33}'}{h} = 0 \tag{IV - 57}$$

$$T_5'(\pm h) = \left(c_{44}' + \frac{e_{35}'^2}{\epsilon_{33}'}\right) k\cosh kh - \frac{e_{35}'^2}{\epsilon_{33}'} \frac{\sinh kh}{h} \quad B$$
$$\left(c_{35}' + \frac{e_{33}' e_{35}'}{\epsilon_{33}'}\right) k\cosh kh - \frac{e_{33}' e_{35}}{\epsilon_{33}'} \frac{\sinh kh}{h} \quad D + \frac{\phi_0 e_{35}'}{h} = 0 \tag{IV - 58}$$

These simultaneous non-homogeneous equations will give an infinite amplitude B and D if the determinant of the coefficients of B and D equals zero (and will thereby specify resonance). This zero determinant becomes after simplification

$$\tanh kh = \frac{kh \left[c_{33}'^{*} c_{55}'^{*} - (c_{35}'^{*2})\right] \epsilon_{33}'}{(c_{33}'^{*} e_{35}'^2 + c_{55}'^{*} e_{33}'^2 - 2c_{35}'^{*} e_{33}' e_{35}')} \tag{IV - 59}$$

Equation (IV - 59) simplifies at θ = zero degrees to the resonance requirement (III - 68) and for $\theta = 90$ degrees becomes

$$\tanh = kh\left[1 + \frac{c_{44}\epsilon_{11}}{e_{15}^2}\right] \qquad \text{(IV - 60)}$$

which corresponds to resonance for the piezoelectrically stiffened shear wave which exists along the x axis.

For a given angle of rotation, the constants of (IV - 59) will be specified by (IV - 46,47,48,49,50,51) and the resonant kh will be specified then by the roots of (IV - 59). The resonant angular velocity will be given by $\omega = k\left[\frac{\lambda}{\rho}\right]^{\frac{1}{2}}$ where the constants λ are the roots to (IV - 45).

D. Additional Modes in a Rotated Plate

There are, in addition to the assumptions (IV - 37,38), two other possibilities which we will see satisfy the boundary conditions (IV - 54) also, and specify a slightly different resonant point. These are

$$u' = F\cos kz'\exp(-i\omega t) \qquad \text{(IV - 61)}$$

$$w' = G\sin kz'\exp(-i\omega t) \qquad \text{(IV - 62)}$$

and

$$u' = H\sin kz'\exp(-i\omega t) \qquad \text{(IV - 63)}$$

$$w' = J\cos kz'\exp(-i\omega t) \qquad \text{(IV - 64)}$$

Considering first the assumptions (IV - 61,62), we find that by substitution into (IV - 36), we have

$$\varphi' = \frac{e'_{35}}{\epsilon'_{33}} F\cos kz' + \frac{e'_{33}}{\epsilon'_{33}} G\sin kz' + c_1 z' + c_2 \qquad \text{(IV - 65)}$$

Applying boundary condition (IV - 40) gives

$$\varphi' = \frac{e'_{35}}{\epsilon'_{33}} F(\cos kz' - \cos kh) + \frac{e'_{33}}{\epsilon'_{33}} G\left(\sin kz' - \frac{\sin kh}{h} z'\right) + \frac{\varphi_0 z'}{h} \qquad \text{(IV - 66)}$$

Proceeding as before, substitutions of (IV - 61,62,66) into the equations (IV - 34,35) gives

$$\left[\left(c'_{55} + \frac{e'^2_{35}}{\epsilon'_{33}}\right)k - \rho\omega^2\right]\cos kz' F + \left(c'_{35} + \frac{e'_{33}e'_{35}}{\epsilon'_{33}}\right)k^2 \sin kz' G = 0 \qquad \text{(IV - 67)}$$

$$\left(c'_{35} + \frac{e'_{33}e_{35}}{\epsilon'_{33}}\right)k^2 \cos kz' F + \left[\left(c'_{33} + \frac{e'^2_{33}}{\epsilon'_{33}}\right)k^2 - \rho\omega^2\right]\sin kz' G = 0 \qquad \text{(IV - 68)}$$

which requires for a non-trivial solution that we satisfy (IV - 45). Substitution in the stress-free boundary condition (IV - 54) gives

$$c'^*_{35} kh \sin kh \, F + \left[c'^*_{33} kh \cos kh - \frac{e'^2_{33}}{\epsilon'_{33}} \sin kh\right] G = -e_{33}\varphi_0 \qquad \text{(IV - 69)}$$

$$c'^*_{55} kh \sin kh \, F + \left[c'^*_{35} kh \cos kh - \frac{e'_{33}e'_{35}}{\epsilon'_{33}} \sin kh\right] G = -e_{35}\varphi_0 \qquad \text{(IV - 70)}$$

which after simplification gives as a resonance condition that

$$\tan kh = kh \frac{\left[c'^*_{33} c'^*_{55} - (c'^*_{35})^2\right] \epsilon'_{33}}{\left(c'^*_{55} e'^2_{33} - c^*_{35} e'_{33} e'_{35}\right)} \qquad \text{(IV - 71)}$$

In the limit of Θ = zero equation (IV - 71) reduces to the equation (III - 68) corresponding to the thickness longitudinal resonance of the unrotated plate. In the limit of $\Theta = 90^{\circ}$ the right-hand side of (IV - 71) becomes infinite, which gives roots for $kh = (2n-1)\pi/2$ and corresponds to the resonance for the elastic longitudinal mode possible on the x axis, so that (IV - 71) specifies resonance for the predominantly longitudinal mode which is of the form (IV - 61,62).

In a similar fashion, the assumptions (IV - 63,64) will be seen to require an electric potential to satisfy the boundary conditions (IV - 40) of the form

$$\frac{e'_{35}}{\epsilon'_{33}}\left(\sin kz' - \frac{\sin khz'}{h}\right)H$$
$$\frac{e'_{33}}{\epsilon'_{33}}(\cos kz' - \cos kh)J + \frac{\varphi_0 z'}{h} \qquad \text{(IV - 72)}$$

which when substituted, along with (IV - 63,64) into (IV - 34,35) gives the same equation (IV - 45) for the velocities. The stress-free requirements (IV - 54) now become

$$\left[c'^{*}_{35}\, kh\cos kh - \frac{e'_{33}e'_{35}}{\epsilon'_{33}}\sin kh\right] H$$
$$- c'^{*}_{33}\, kh\sin kh\, J = -\varphi_0 e'_{33} \qquad \text{(IV - 73)}$$
$$\left[c'^{*}_{55}\, kh\cos kh - \frac{e'^{2}_{35}}{\epsilon'_{33}}\sin kh\right] H$$
$$- c'^{*}_{35}\, kh\sin kh\, J = -\varphi_0 e'_{35} \qquad \text{(IV - 74)}$$

which give as a resonance requirement

$$\tan kh = kh\,\frac{\left[c'^{*}_{33}c'^{*}_{55} - (c'^{*}_{35})^2\right]\epsilon'_{33}}{(c'^{*}_{33}e'^{2}_{35} - c'_{35}e'_{33}e'_{35})} \qquad \text{(IV - 75)}$$

In the limit $\Theta =$ zero degrees, the right-hand side of (IV - 75) becomes infinite and the roots are $(2n-1)\pi/2$

which correspond to the elastic shear mode on the z axis, and in the limit $\theta = 90$ degrees (IV - 75) reduces to (IV - 60) which is the resonance requirement for the piezoelectric shear mode which propagates along the x crystallographic axis. It would seem then that just as (IV - 71) corresponded to the resonance of the predominantly longitudinal modes (IV - 75) corresponds to the predominantly shear modes. The three equations (IV - 59,71,75) reduce to the same results for the particular cases of zero and a full ninety degree rotation; however, it is evident that they will specify somewhat different roots at intermediate degrees of rotation. The reason is that we are considering a somewhat different mode of propagation when we specify both displacements u' and w' to be a sine function, than when one is a sine and one a cosine function. The difference lies in the particle motion. Since the u' and w' displacements are always in phase in the former case, we are specifying a particle motion which is along a straight line, the orientation of the line varying between the x and z axis limits depending upon the relative amplitudes. In the latter case, the particle motion is elliptic tending in the two limits of relative amplitude to linear oscillation along the x and z axes.

E. A Plate of Barium Titanate Rotated Forty-five Degrees

The numerical values for barium titanate corresponding to a forty-five degree angle of rotation are:

$c'_{33} = 16.37 \times 10^{10}$, $c'_{35} = \mp 1 \times 10^{10}$, $c'_{55} = 4.32 \times 10^{10}$,
$e'_{33} = 11.8$, $e'_{35} = 6.27$, $\epsilon'_{33} = 110.7 \times 10^{-10}$, $c'^{*}_{33} = 17.6 \times 10^{10}$,
$c'^{*}_{35} = .57 \times 10^{10}$, and $c'^{*}_{55} = 4.68 \times 10^{10}$.

Substituting these values into (IV - 59,71,75) gives tankh = 7.21kh, tankh = 14.9kh, and tankh = 14.0kh respectively. The first roots to these are kh = 1.48, kh = 1.53 and kh = 1.52. The velocity equation (IV - 45) gives the roots $\lambda = \frac{\rho\omega^2}{k^2} = 17.6 \times 10^{10}$ and 4.68×10^{10}. We see then that we can expect the first resonant angular velocity for a 45° plate to occur for the predominantly longitudinal modes at $\omega = \frac{5.61 \times 10^3\ (1.48)}{h}$ or $\omega = \frac{5.61 \times 10^3\ (1.53)}{h}$, and the first resonant angular velocity for predominantly shear modes at $\omega = \frac{2.89 \times 10^3 (1.48)}{h}$ or at $\omega = \frac{2.89 \times 10^3 (1.52)}{h}$. In each case, the determining factor as to which type of particle motion occurs physically will be specified by other factors, such as the satisfaction of a free lateral face.

V. EDGE EFFECTS (a "Finite" plate)

A. The Infinite Plate Solution Near an Edge

We found for the infinite plate that the piezoelectrically generated wave was a thickness longitudinal mode composed of a w (or z) displacement only, which satisfied the boundary condition $T_5(\pm h) = 0$ identically by having no u (or x) displacement, and which satisfied the $T_3(\pm h) = 0$ boundary condition with an infinite amplitude for certain wave numbers. If we now introduce a lateral face parallel to the z axis at $x = 0$ we will now have to satisfy the requirements of a stress-free lateral face, viz.

$$T_1(0) = 0 \qquad \text{(V - 1)}$$

$$T_5(0) = 0 \qquad \text{(V - 2)}$$

Applying our infinite plate solution condition (V - 2) can be satisfied if the u displacement is zero. The non-zero terms of T_1 are, from (III - 25)

$$T_1(0) = c_{13} w_{,z}(0) + e_{31} \varphi_{,z}(0) = 0 \qquad \text{(V - 3)}$$

Substituting our expressions from the infinite plate (III - 59) and (III - 61) into (V - 3), we find

$$c_{13} k \cos kz \, C + \frac{e_{31} e_{33}}{\epsilon_{33}} k \cos kz \, C - \frac{e_{31} e_{33}}{\epsilon_{33}} \frac{\sin kh}{h} C \quad e_{31} \frac{\varphi_0}{h} = 0 \qquad \text{(V - 4)}$$

or

$$\left[\left(c_{13} + \frac{e_{31} e_{33}}{\epsilon_{33}}\right) kh \cos kz - \frac{e_{31} e_{33}}{\epsilon_{33}} \sin kh\right] C = - e_{31} \varphi_0 \qquad \text{(V - 5)}$$

Equation (V - 5) cannot be satisfied along the entire lateral face unless the amplitude C is a function of z. The question arises then as to how good an approximation we have if we satisfy (V - 5) at $z = h$ so as to specify resonance at the upper face where, since we have an anti-symmetric oscillation, the majority of the energy is located. It is a simple matter to substitute the values for barium titanate in the equation (V - 5) for $kh = 1.49$ corresponding to the first resonance of the upper and lower surfaces and thereby find that the variation of C with z is such that $C(h) \cong 5.7\, C(0)$, and that $C_{,z}$ is very large near the edge $z = h$ to the end that our assumption that $w_{,z} = C(\sin kz)_{,z}$ is far from accurate near $z = h$. It will be necessary to examine the implications of a lateral edge more closely.

B. The Lateral Edge

It was shown previously that elastic waves propagating in a hexagonal piezoelectric medium have to satisfy the determinant (III - 37) and whatever boundary conditions are imposed. We want now to examine the pre-requisites for satisfaction of stress-free faces on both the upper and lower, as well as a lateral face. The stress involved are

$$T_1 = c_{11} u_{,x} + c_{13} w_{,z} + e_{31} \varphi_{,z} \qquad \text{(V - 9)}$$

$$T_3 = c_{13} u_{,x} + c_{33} w_{,z} + e_{33} \varphi_{,z} \qquad \text{(V - 10)}$$

$$T_5 = c_{44} u_{,z} + c_{44} w_{,x} + e_{15} \varphi_{,x} \qquad \text{(V - 11)}$$

We are to satisfy $T_1(1) = T_5(1) = 0$ for all values of z along some lateral edge designated by the plane $x = 1$. More specifically, we are to satisfy the boundary conditions

$$T_1(x = 1) = T_5(x = 1) = 0 \qquad (V - 12)$$

$$T_3(z = \pm h) = T_5(z = \pm h) = 0 \qquad (V - 13)$$

Assume a solution to (V - 9,10,11) which is of the form

$$u = A \exp i(k_1 x + k_3 z - \omega t) + B \exp i(k_1 x - k_3 z - \omega t) \qquad (V - 14)$$

$$w = C \exp i(k_1 x + k_3 z - \omega t) + D \exp i(k_1 x - k_3 z - \omega t) \qquad (V - 15)$$

$$\varphi = F \exp i(k_1 x + k_3 z - \omega t) + G \exp i(k_1 x - k_3 z - \omega t) \qquad (V - 16)$$

We have speculated here that standing waves are generated only in the z direction. Substitution of (V - 14, 15, 16) into the boundary conditions (V - 12, 13) will give us six homogeneous equations in the six unknown constants A,B,C,D,F,G. Rather than write all six equations out in full, we will consider first only three of these, the three corresponding to the requirements

$$T_5(x = 1) = T_5(z = \pm h) = 0 \qquad (V - 17)$$

Writing these three in their entirety, we have

$$T_5(x = 1) = c_{44} k_3 \exp i(k_1 l + k_3 z)A - c_{44} k_3 \exp i(k_1 l - k_3 z)B$$
$$c_{44} k_1 \exp i(k_1 l + k_3 z)C + c_{44} k_1 \exp i(k_1 l - k_3 z)D$$
$$+ e_{15} k_1 \exp i(k_1 l + k_3 z)F + e_{15} k_1 \exp i(k_1 l - k_3 z)G = 0 \qquad (V - 18)$$

$$T_5(z=+h) = c_{44}k_3 \exp i(k_1x + k_3h)A - c_{44}k_3 \exp i(k_1x-k_3h)B$$
$$+c_{44}k_1 \exp i(k_1x+k_3h)C + c_{44}k_1 \exp i(k_1x-k_3h)D$$
$$+e_{15}k_1 \exp i(k_1x+k_3h)F + e_{15}k_1 \exp i(k_1x-k_3h)G = 0 \quad (V-19)$$
$$T_5(z=-h) = c_{44}k_3 \exp i(k_1x-k_3h)A - c_{44}k_3 \exp i(k_1x+k_3h)B$$
$$c_{44}k_1 \exp i(k_1x-k_3h)C + c_{44}k_1 \exp i(k_1x+k_3h)D$$
$$e_{15}k_1 \exp i(k_1x-k_3h)F + e_{15}k_1 \exp i(k_1x+k_3h)G = 0 \quad (V-20)$$

Multiplying (V - 20) by $\exp i(2k_3h)$ and subtracting the result from (V - 19) gives, after cancelling common terms

$$T_5(z=+h) = 0\,A - c_{44}k_3\,B + 0\,C + c_{44}k_1\,D + 0\,F + e_{15}k_1G = 0 \quad (V-21)$$

Multiplying (V - 21) by $\exp i(k_1x+k_3h)$ and subtracting the result from (V - 20) and then multiplying (V - 21) by $\exp i(k_1l-k_3z)$ and subtracting the result from (V - 18) and cancelling any common terms in both cases gives us

$$T_5(x=l) = c_{44}k_3A + 0\,B + c_{44}k_1C + 0\,D + e_{15}k_1F + 0\,G = 0 \quad (V-22)$$
$$T_5(z=-h) = c_{44}k_3A + 0\,B + c_{44}k_1C + 0\,D + e_{15}k_1F + 0\,G = 0 \quad (V-23)$$

We now have two of the six homogeneous equations which are identical. As a result, the determinant of the coefficients of the six constants will be zero, assuring a non-trivial solution. The result that a non-trivial solution of the form (V - 14, 15, 16) exists, regardless of the dimensions $x=l$, $z=\pm h$ implies that the stresses T_1, T_3

and T_5 can be satisfied everywhere by displacements of the (V - 14,15,16) type.

Continuing in this manner, assume now displacements of the form

$$u = A \exp i(k_1 x) \cos k_3 z \qquad (V - 24)$$

$$w = B \exp i(k_1 x) \sin k_3 z \qquad (V - 25)$$

$$\varphi = C \exp i(k_1 x) \sin k_3 z \qquad (V - 26)$$

where we are going to consider satisfaction of the stresses T_1 T_3 and T_5 everywhere. Substitution of (V - 24, 25, 26) into (V - 9,10,11) gives

$$T_1 = c_{11} ik_1 \exp i(k_1 x)\cos k_3 zA + c_{13} k_3 \exp i(k_1 x)\cos k_3 zB + e_{31} k_3 \exp i(k_1 x)\cos k_3 zC = 0 \qquad (V - 27)$$

$$T_3 = c_{13} ik_1 \exp i(k_1 x)\cos k_3 zA + c_{33} k_3 \exp i(k_1 x)\cos k_3 zB + e_{33} k_3 \exp i(k_1 x)\cos k_3 zC = 0 \qquad (V - 28)$$

$$T_5 = -c_{44} k_3 \exp i(k_1 x)\sin k_3 zA + c_{44} ik_1 \exp i(k_1 x)\sin k_3 zB + e_{15} ik_1 \exp i(k_1 x)\sin k_3 zC = 0 \qquad (V - 29)$$

The determinant of the coefficients equated to zero gives

$$(c_{11} c_{33} e_{15} + c_{15} c_{44} e_{31} - e_{15} c_{13}^2 - c_{11} c_{44} e_{33}) k_1^2 = c_{44}(e_{31} c_{33} - e_{33} c_{13}) k_3^2 \qquad (V - 30)$$

A substitution of the values for barium titanate into (V - 30) gives $k_1 = 1.16 i k_3$. The implication then is that since k_1 in (V - 24,25,26) is imaginary, the exponential term becomes $\exp - 1.16k_3x$ specifying an exponential decrease of amplitude from the lateral edge. The magnitude of this decrease is consistent with the approximate limit for infinite plate theory of a width of three and a half times the thickness.(19) A plate resonant at $k_3h = 1.5$ will have an edge effect which has diminished to ¼% of its edge amplitude at a distance of 1.75 times the thickness, or mid-point of a plate which is three and a half times wider than its thickness.

Substitution of the value $k_1 = 1.16ik_3$ along with the other values for barium titanate into the determinant (III - 37) gives us a numeric value for $\lambda = \frac{\rho\omega^2}{k^2}$ which is a complex number, viz. $\frac{\rho\omega^2}{k^2} = -3.06 \pm 2.95i = 4.25 \text{ expi}(.756\pi, 1.25\pi)$ or $\frac{\omega}{k} = 2.76 \times 10^3 \text{ expi}(.378\pi, .622\pi, 1.37\pi, 1.56\pi)$.

VI. SUMMARY

It was shown in Section III that elastic waves propagate in a piezoelectric medium of hexagonal symmetry with a velocity which is symmetric about the z axis, and otherwise directionally dependent. The velocity profile of barium titanate was given in Table II. An assumption that the electric potential was a function of only the z coordinate resulted in a requirement that the displacements, as well, be a function of only the z coordinate. Considering only z dependent functions, the solution to the infinite plate was found to involve a piezoelectrically stiffened thickness longitudinal mode which experienced resonance at angular velocities which were not integral multiples of the first, but rather the roots to (III - 68).

A "rotated" system was shown to involve a number of extra terms in the wave equations; however, once again the restriction that the electric potential be a function of the z', or rotated z, coordinate gives displacements which are similarly functions of the z' coordinate only. It was shown that the satisfaction of the wave equations and the boundary conditions of stress-free plate faces requires both the longitudinal and the shear modes in a rotated system.

Two types of solution were considered, those such as (IV - 37,38) in which the u' and w' displacements are precisely in phase, and which specify a planar particle motion, and those such as (IV - 61,62,63,64) in which the u' and w' displacements are not in phase and which permit an elliptic particle motion.

These two types are indistinguishable for a normal or fully rotated plate; however, for intermediate angles of rotation, each type specifies a slightly different mode and a slightly different resonant frequency.

It was shown in Section V that the solutions to the infinite plate were not enough to formally satisfy the boundary conditions for a stress-free lateral face. A piezoelectric plate with a single lateral face was considered for displacements of the (V - 14,15,16) type which specified a standing wave, in the z direction. Substitution of these displacements into the six boundary conditions necessary to specify a stress-free upper, lower, and lateral surface gives six homogeneous equations in the six unknown amplitudes. Manipulation of these equations will make one equation entirely zero, so that a non-trivial solution is always guaranteed for any plate dimensions, implying that the stresses can be satisfied everywhere by displacements of the assumed type. Satisfaction of the stresses everywhere gives a standing wave in the z direction modulated by an exponentially decreasing x-dependent amplitude.

It has been experimentally observed that resonance for finite plates which are less than three and a half times as wide as their thickness begin to depart seriously from their "infinite" plate resonant frequencies.(19) The numeric value for barium titanate of the x-dependence of the edge associated modes is consistent with this observed lower limit for an "infinite" plate of three and a half times its thickness.

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